# 2DH Curves

Quadratic

# The Quadratic Curve Equation

$$Ax^2 + 2Bxy + Cy^2$$
$$+ 2Dxw + 2Eyw$$
$$+ Fw^2 = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \mathbf{p}\mathbf{Q}\mathbf{p}^T = 0$$

$$\boxed{p \rightarrow Q \leftarrow p} = 0$$

# Transform to

$$\mathbf{T}^{*}\begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix}\mathbf{T}^{*T} = \begin{bmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_w \end{bmatrix}$$

$$\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_w \end{bmatrix}\begin{bmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_w \end{bmatrix}\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_w \end{bmatrix} = \begin{bmatrix} U_x & 0 & 0 \\ 0 & U_y & 0 \\ 0 & 0 & U_w \end{bmatrix}$$

$$U_i = -1, 0, +1$$

# The Catalog

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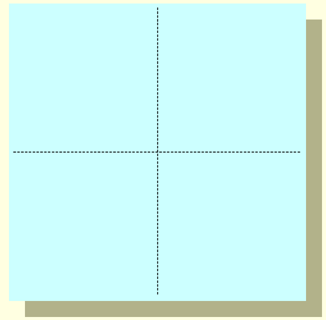
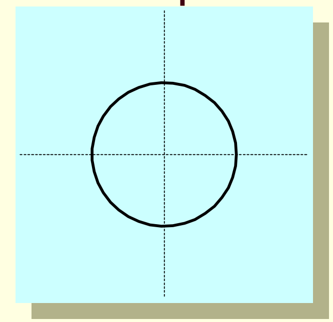


$x^2+y^2+w^2=0$

++-

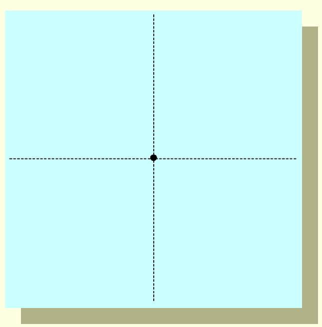
--+



$x^2+y^2-w^2=0$

++0

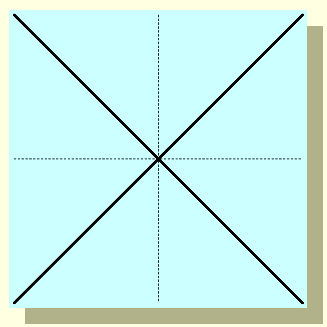
--0



$x^2+y^2=0$

+-0

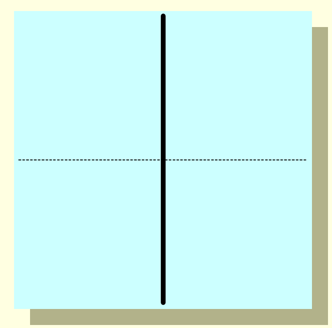
-+0



$x^2-y^2=0$

+00

-00

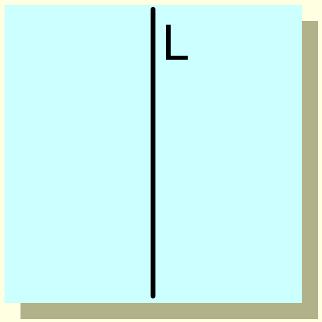


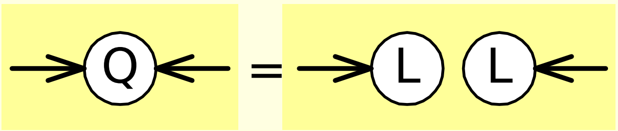
$x^2=0$

# Analysis/Synthesis of Forms

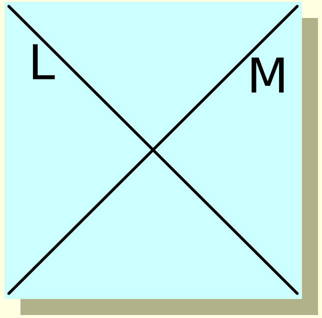
- Detect Which Type
- Construct Desired Type from Geometric Info
- Deconstruct Known Type into Geometric Info
- Stationary Transforms

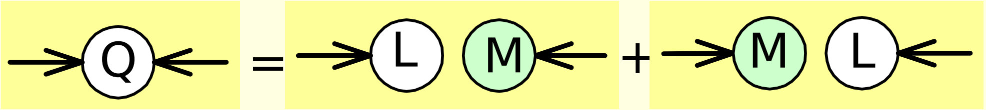
# Reducible Quadratics



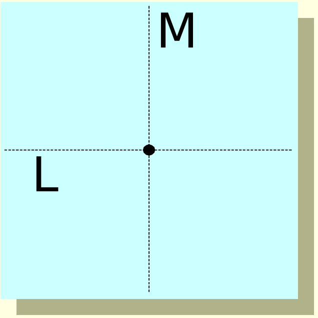



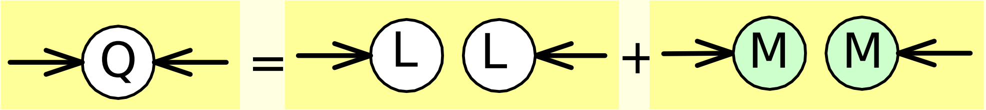

$$\mathbf{PQP}^T = \left(\mathbf{P}\times\mathbf{L}\right)^2$$



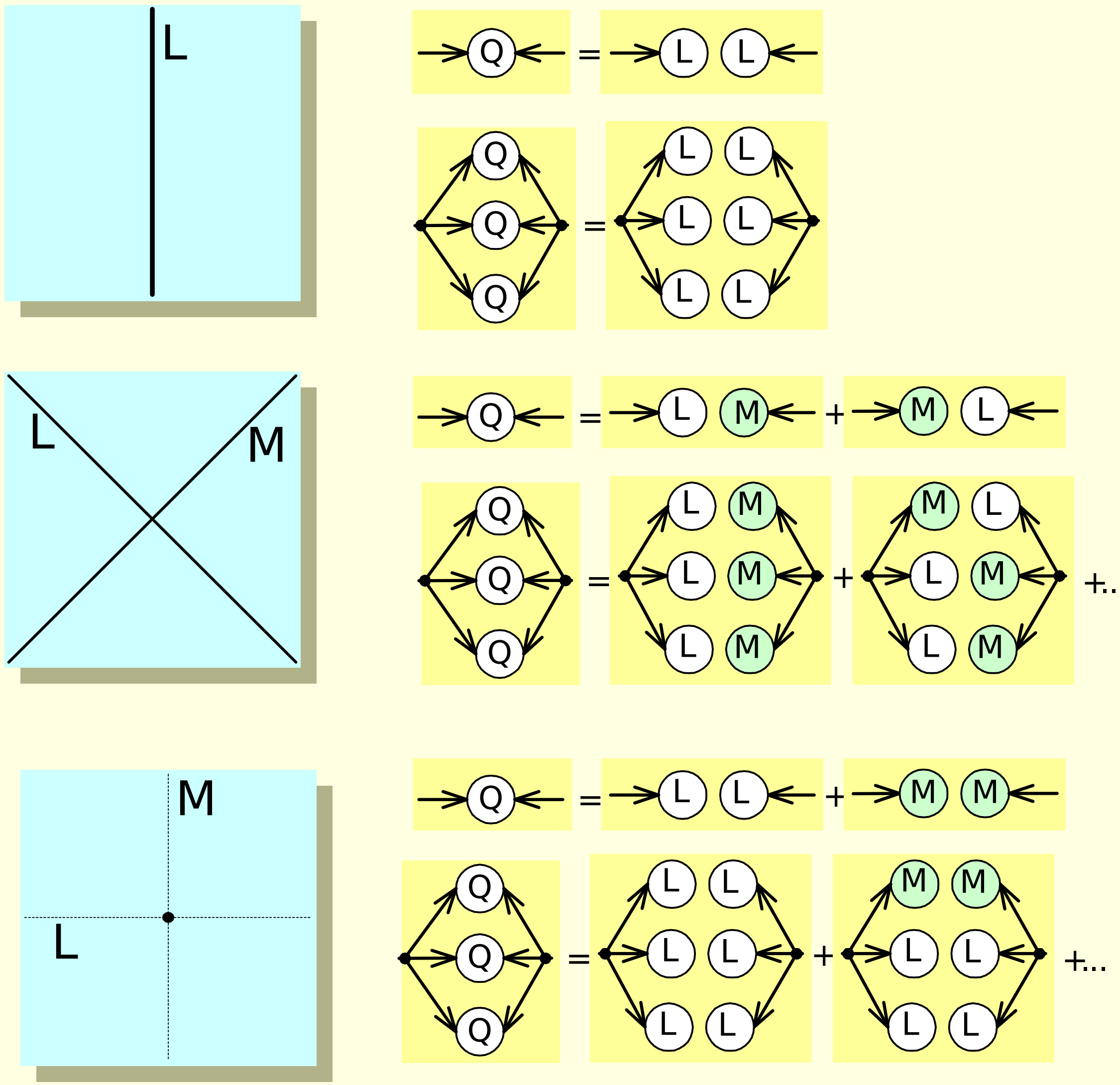

$$\mathbf{PQP}^T = 2\left(\mathbf{P}\times\mathbf{L}\right)\left(\mathbf{P}\times\mathbf{M}\right)$$



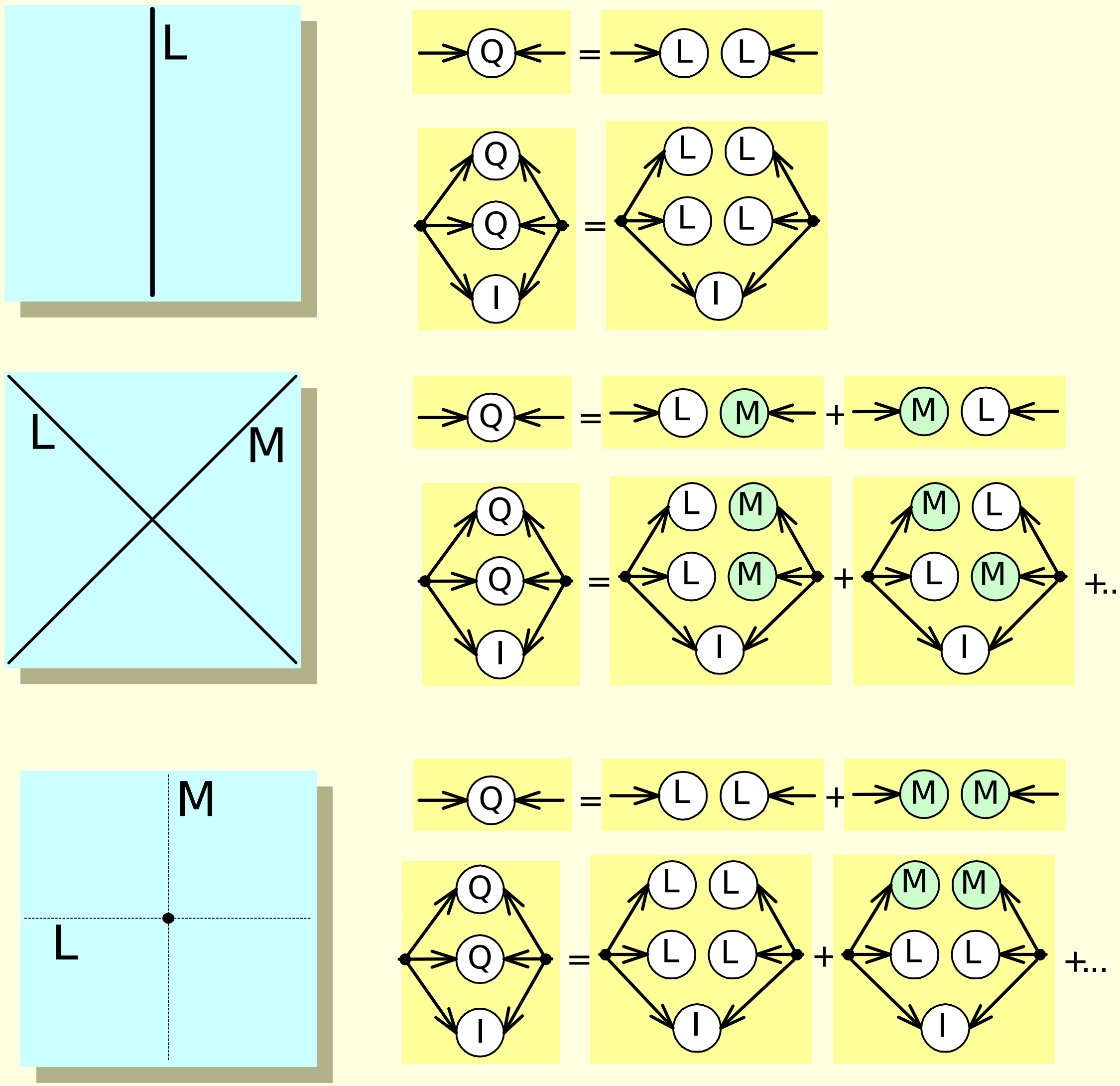




$$\mathbf{PQP}^T = \left(\mathbf{P}\times\mathbf{L}\right)^2 + \left(\mathbf{P}\times\mathbf{M}\right)^2$$

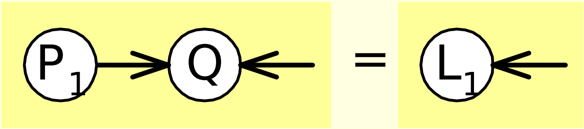
# Determinant of Reducible Q

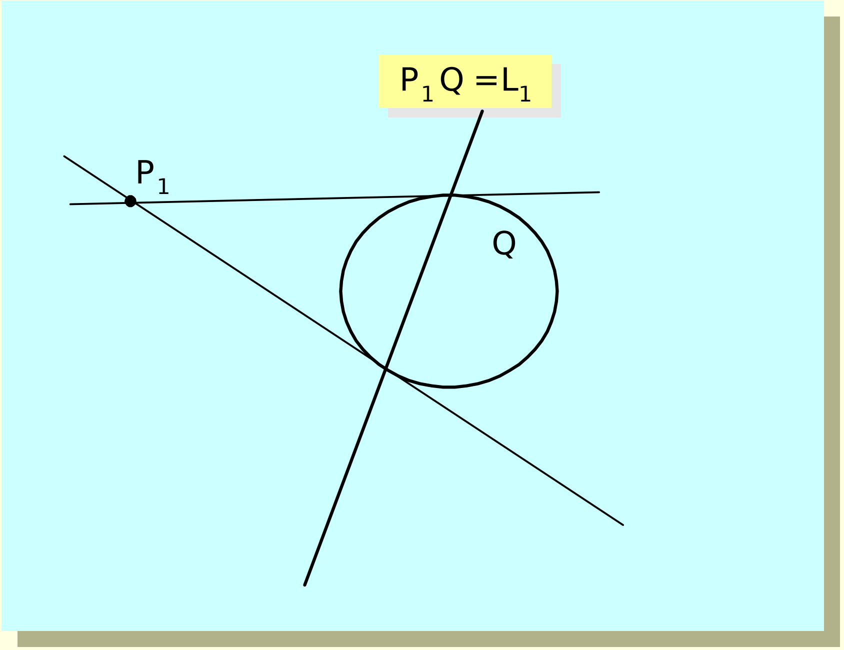


# TraceAdjoint of Reducible Q

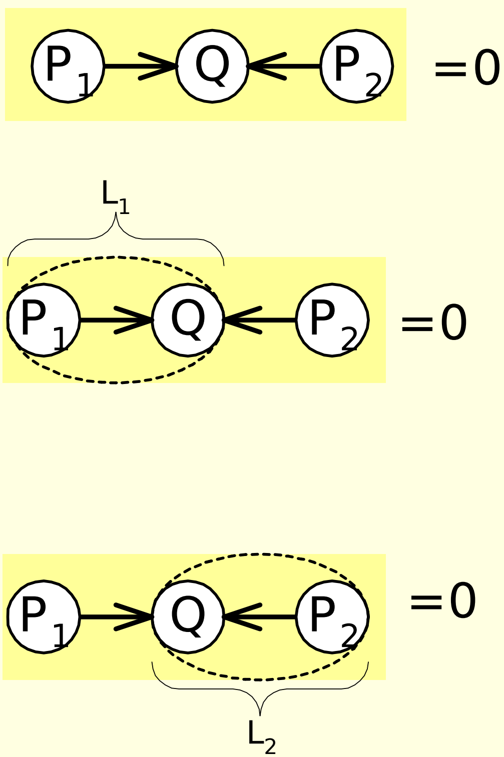


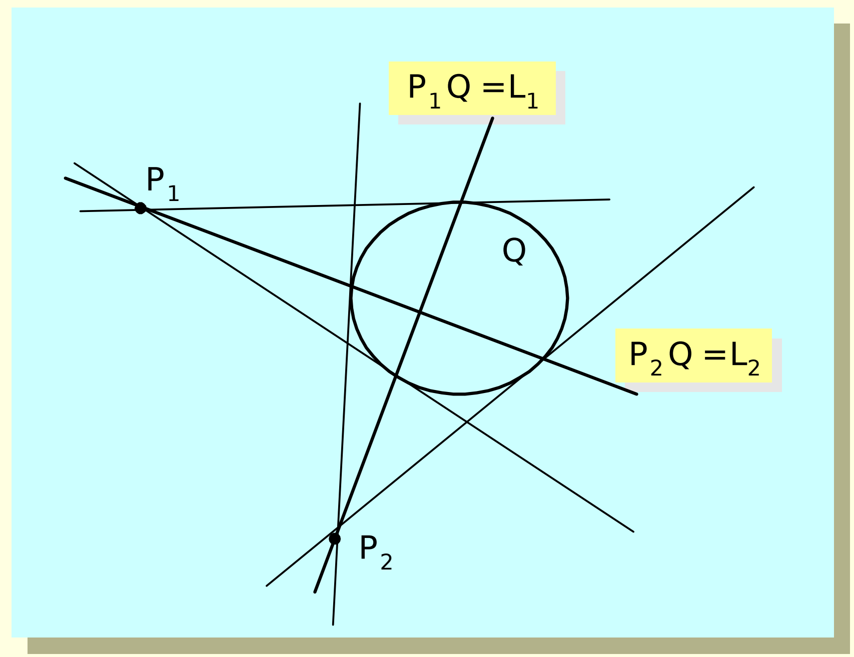
# Conic Sections and Polars



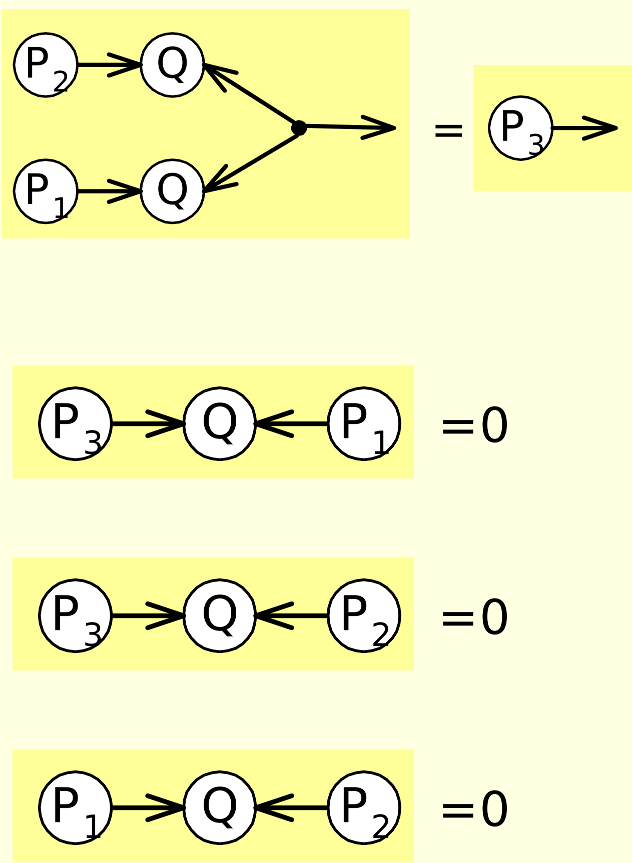



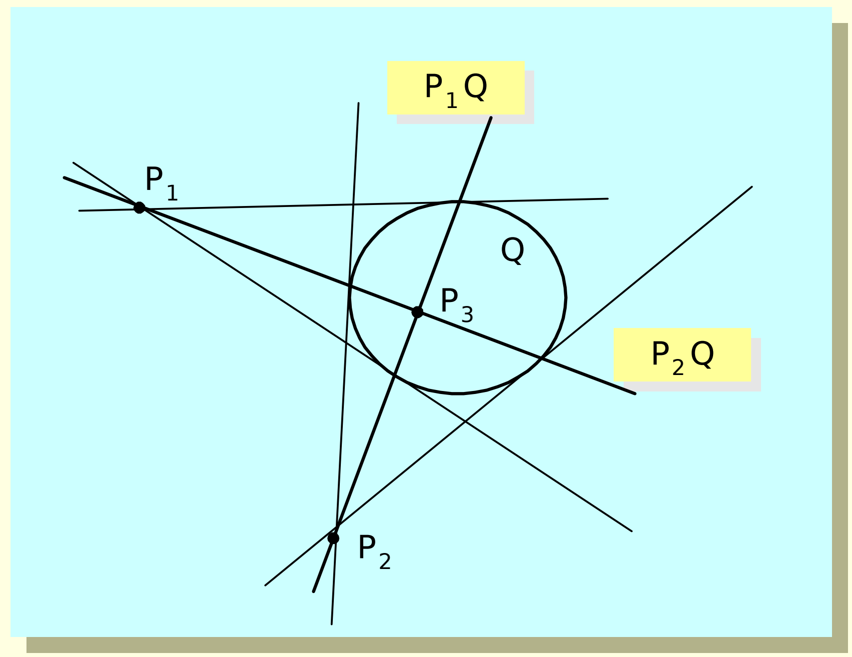
# Second Polar



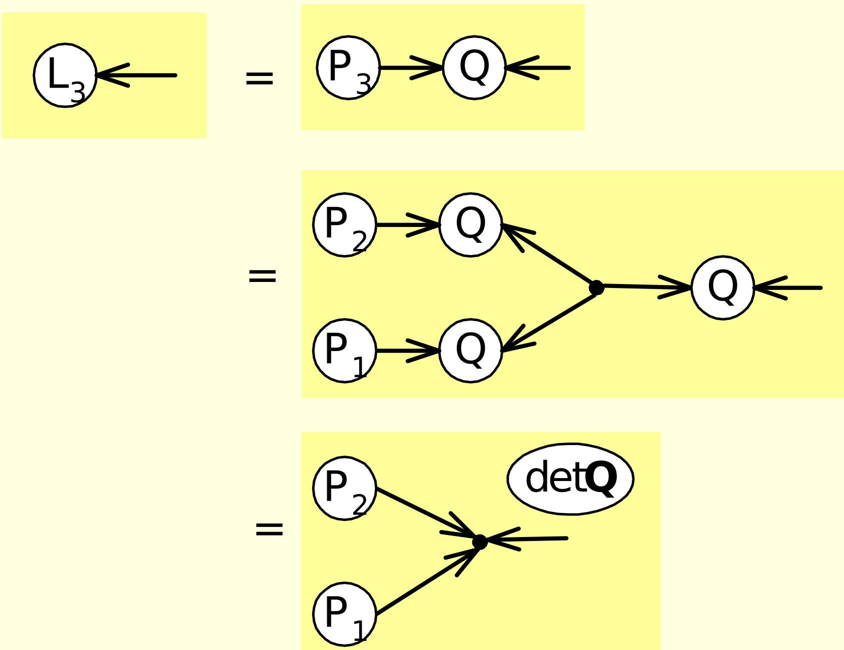



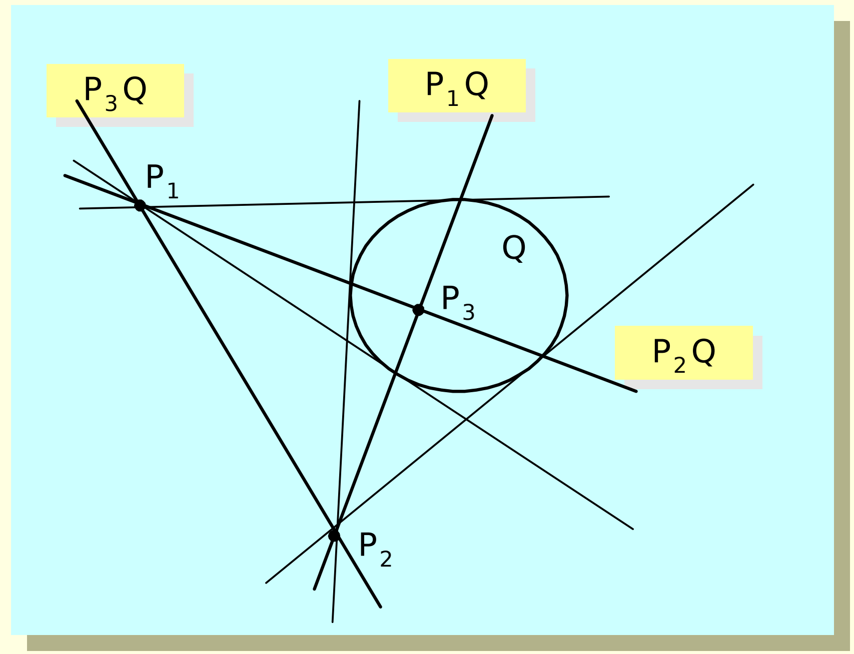
# Third Polar



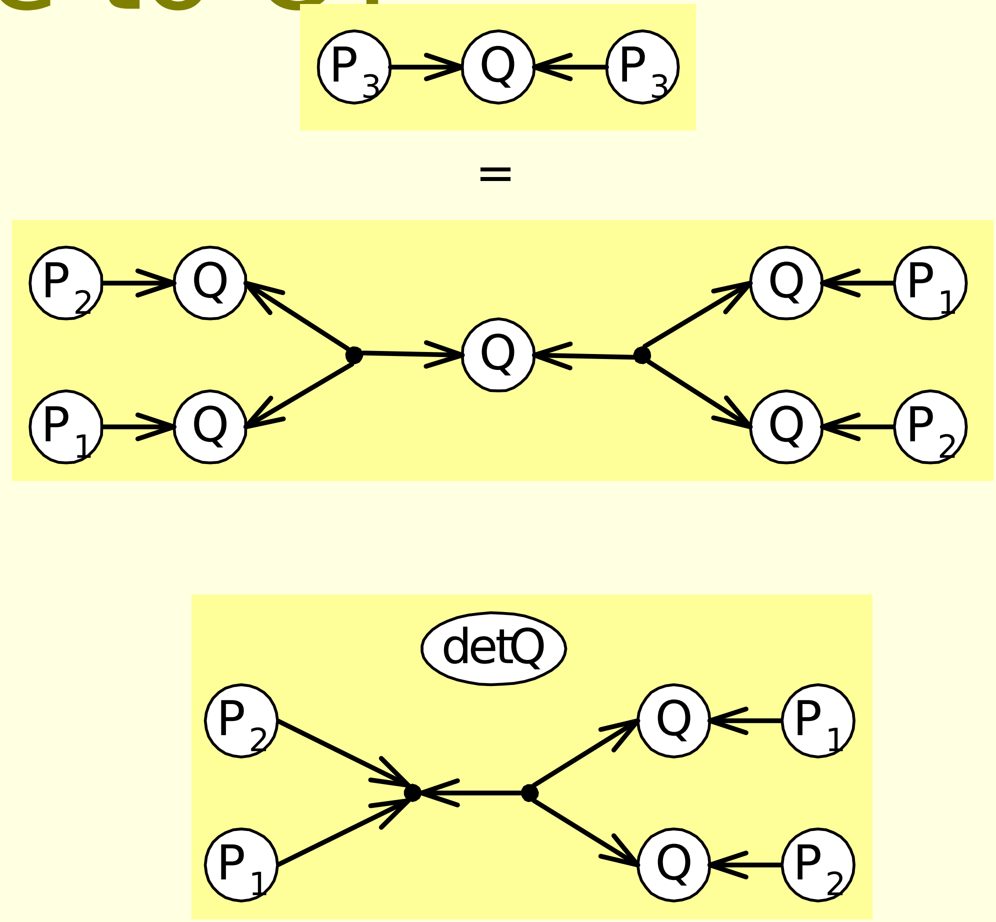



# Polar Line To P3

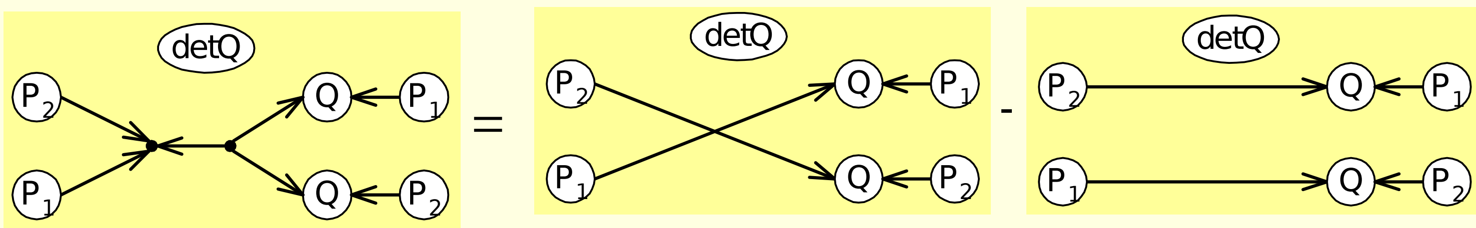


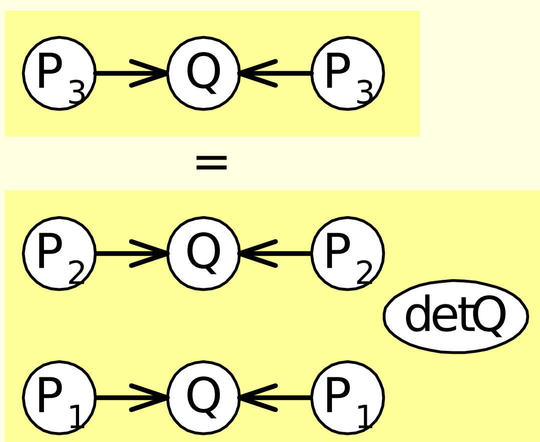



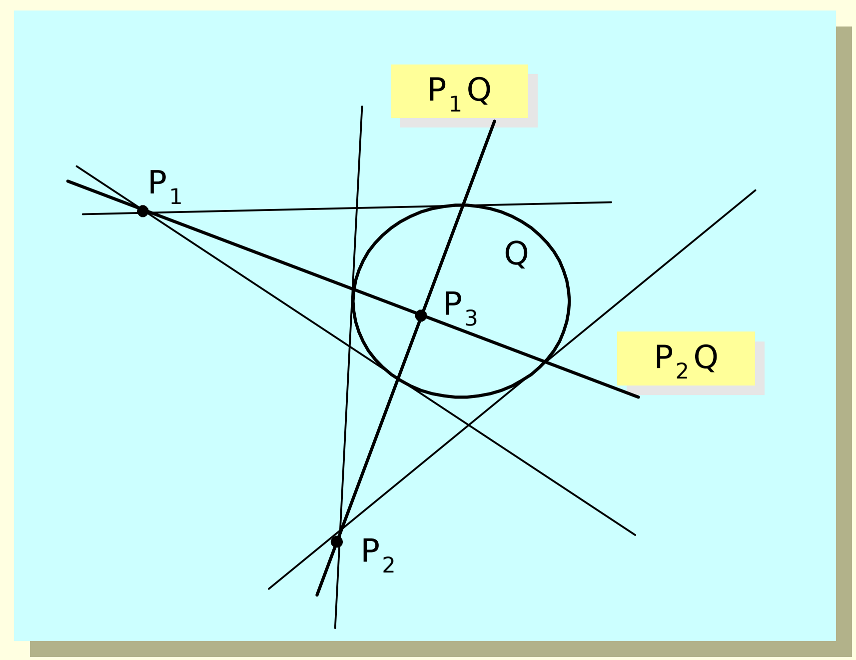
# How Does Third Polar Relate to Q?



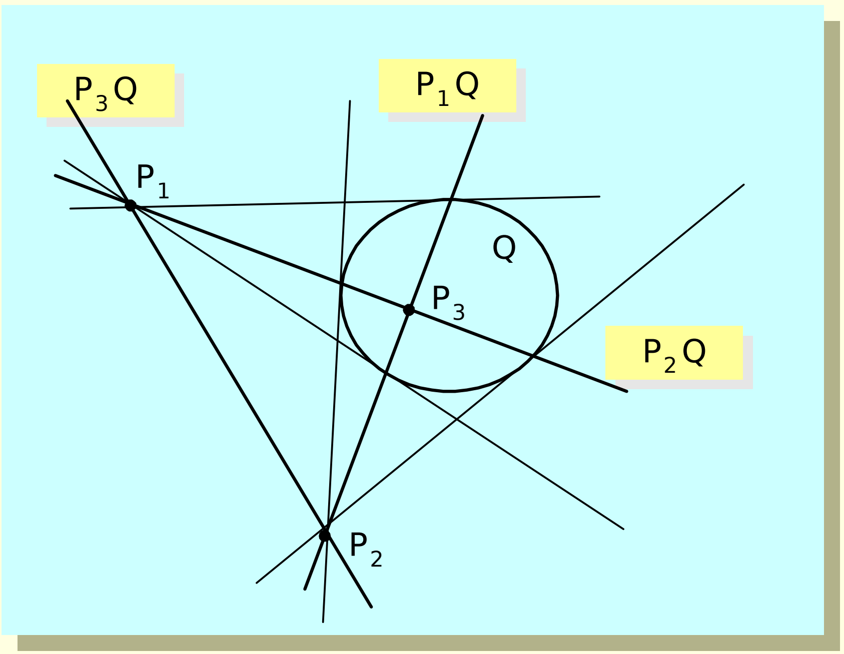
# P3 Relation to Q



# Make A Transformation Out Of P1,P2,P3

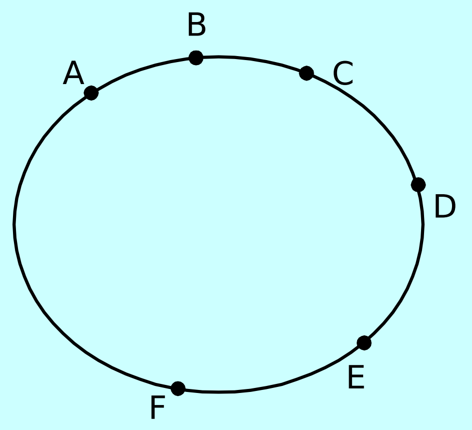



$$\begin{bmatrix} \cdots & \mathbf{P}_1 & \cdots \\ \cdots & \mathbf{P}_2 & \cdots \\ \cdots & \mathbf{P}_3 & \cdots \end{bmatrix} \mathbf{Q} \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3 \\ \vdots & \vdots & \vdots \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{P}_1\mathbf{Q}\mathbf{P}_1^T & \mathbf{P}_1\mathbf{Q}\mathbf{P}_2^T & \mathbf{P}_1\mathbf{Q}\mathbf{P}_3^T \\ \mathbf{P}_2\mathbf{Q}\mathbf{P}_1^T & \mathbf{P}_2\mathbf{Q}\mathbf{P}_2^T & \mathbf{P}_2\mathbf{Q}\mathbf{P}_3^T \\ \mathbf{P}_3\mathbf{Q}\mathbf{P}_1^T & \mathbf{P}_3\mathbf{Q}\mathbf{P}_2^T & \mathbf{P}_3\mathbf{Q}\mathbf{P}_3^T \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{P}_1\mathbf{Q}\mathbf{P}_1^T & 0 & 0 \\ 0 & \mathbf{P}_2\mathbf{Q}\mathbf{P}_2^T & 0 \\ 0 & 0 & \mathbf{P}_3\mathbf{Q}\mathbf{P}_3^T \end{bmatrix}$$

# Pascal's (Pappus') Theorem



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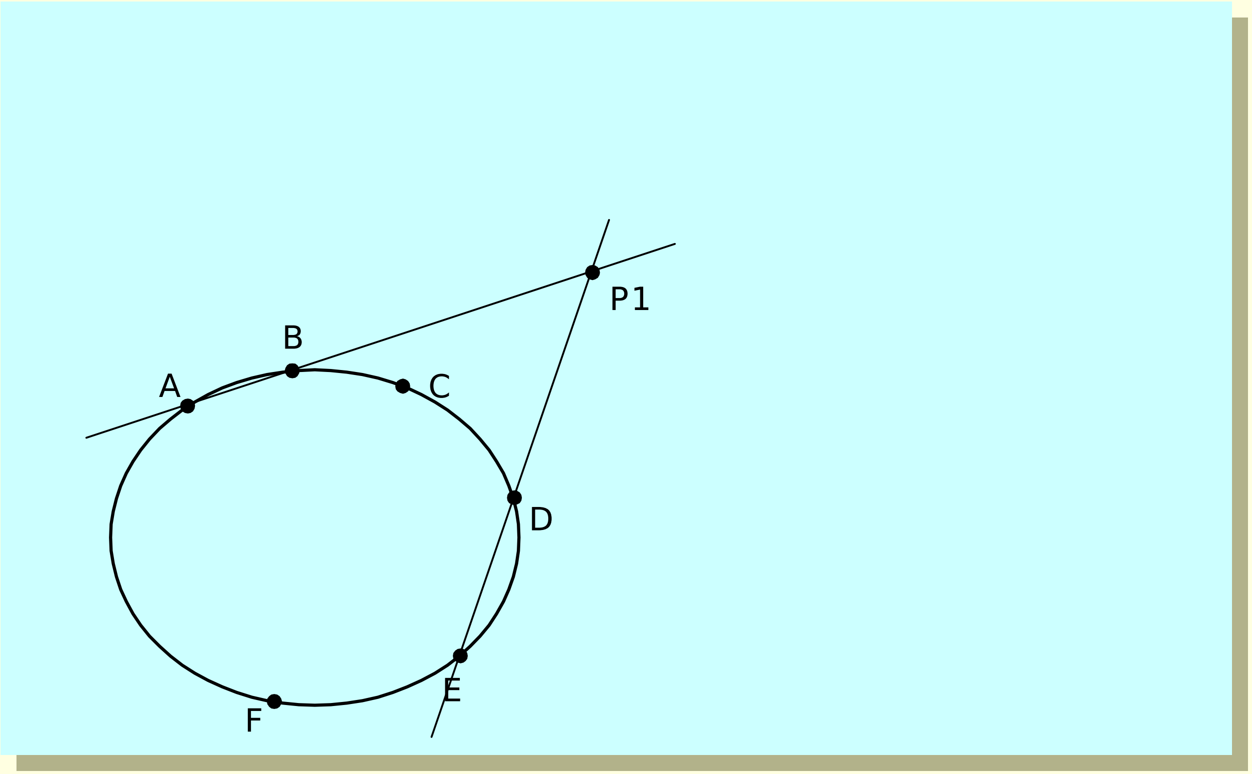

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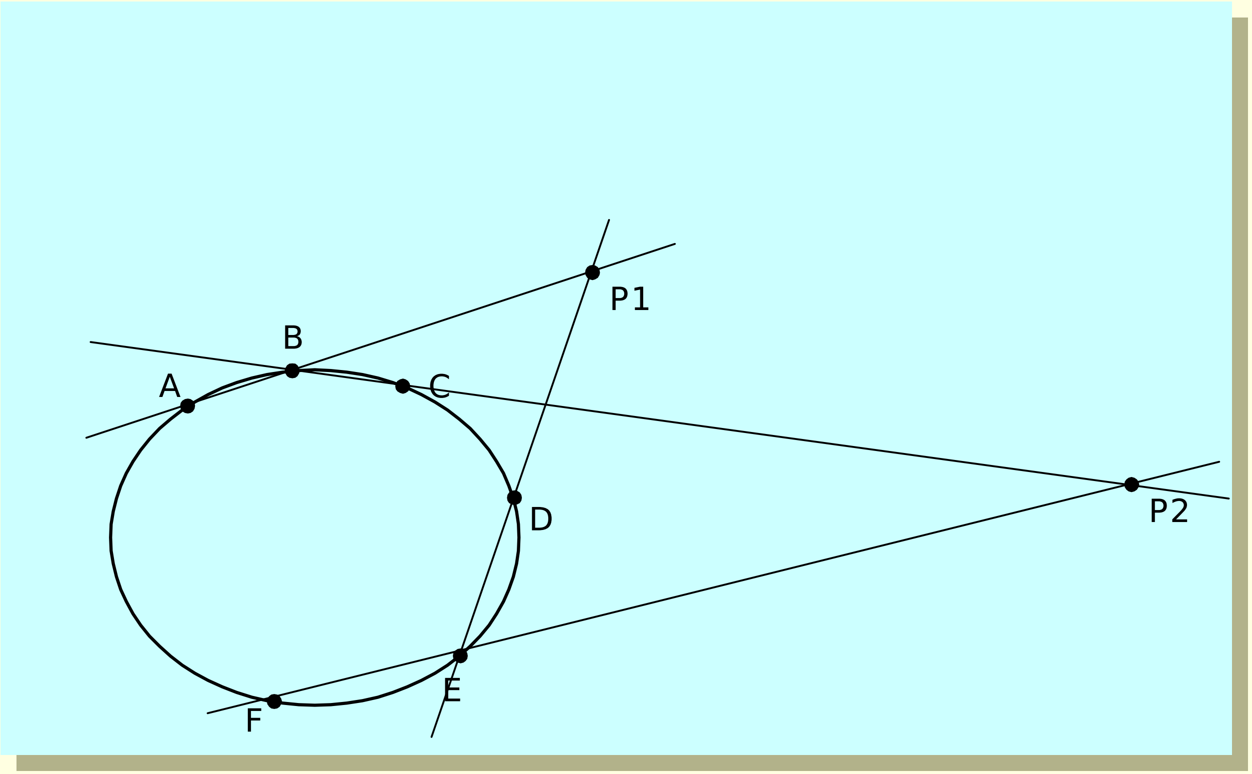

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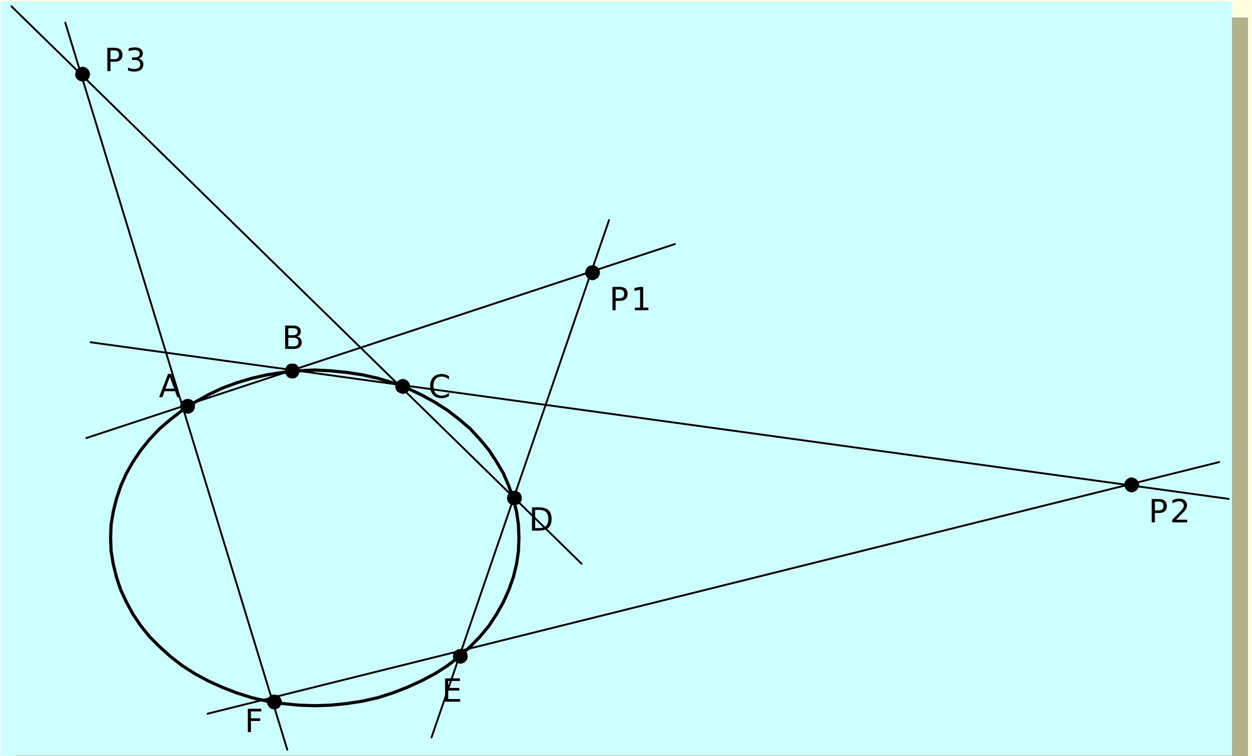

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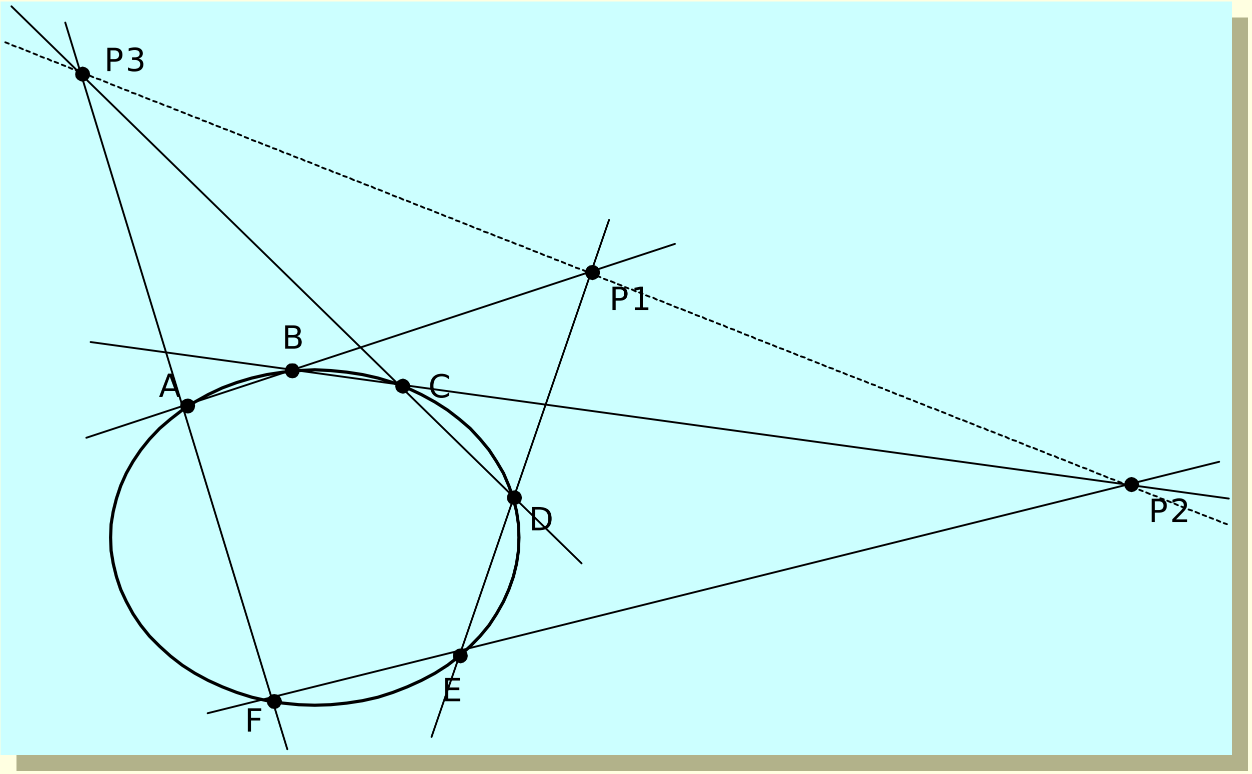

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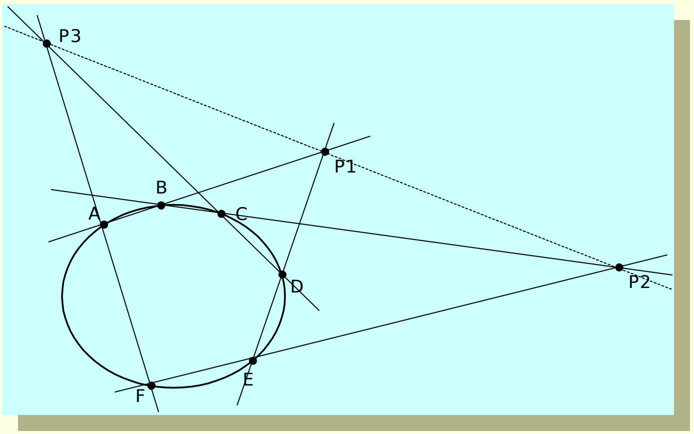
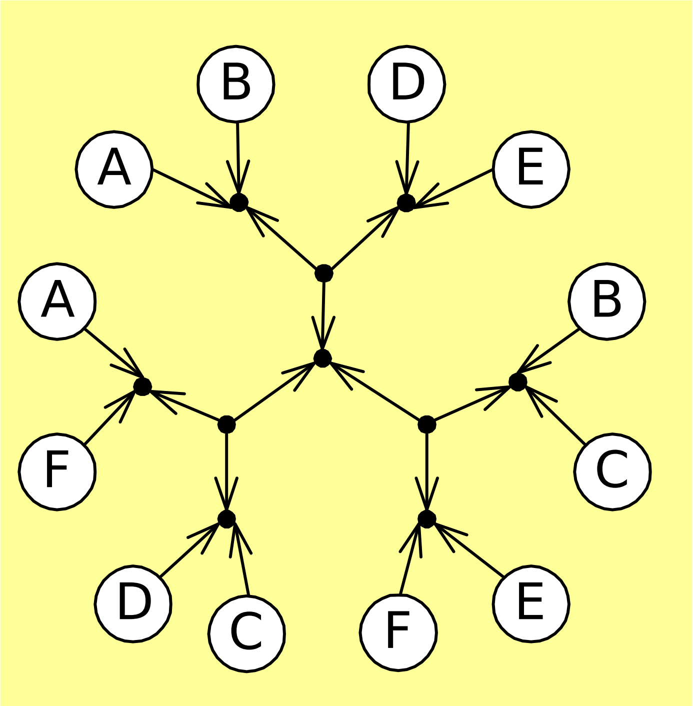


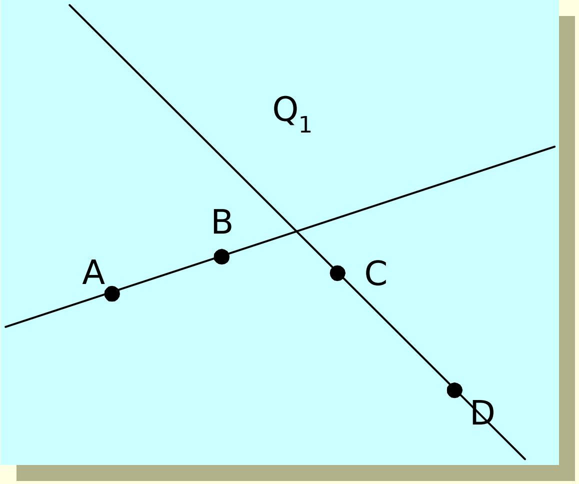

=0

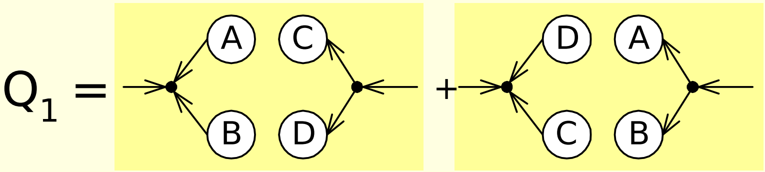
# Conic Section on 5 Points

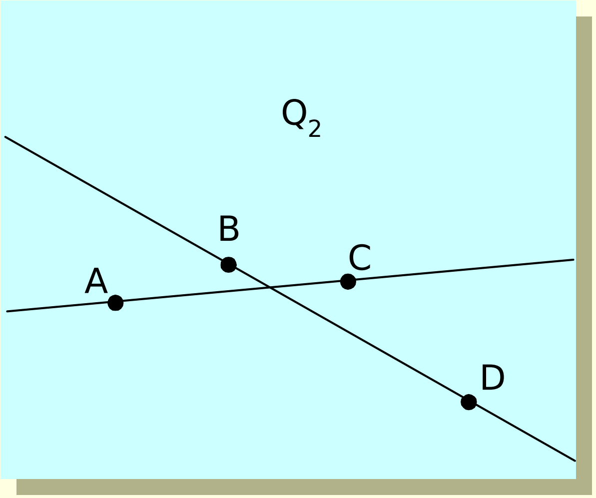
$$Ax^2 + 2Bxy + Cy^2 + 2Dxw + 2Eyw + Fw^2 = 0$$

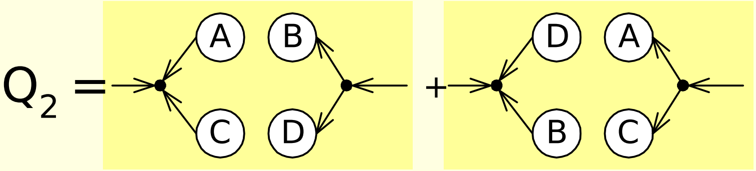
$$\begin{bmatrix} x_1^2 & 2x_1y_1 & y_1^2 & 2x_1w_1 & 2y_1w_1 & w_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_5^2 & 2x_5y_5 & y_5^2 & 2x_5w_5 & 2y_5w_5 & w_5^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# A (Better) Way

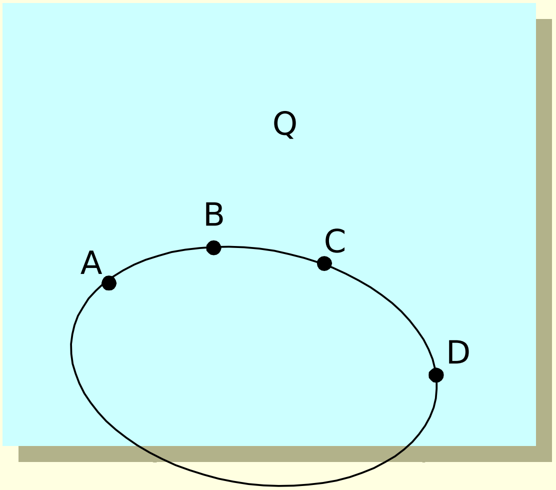


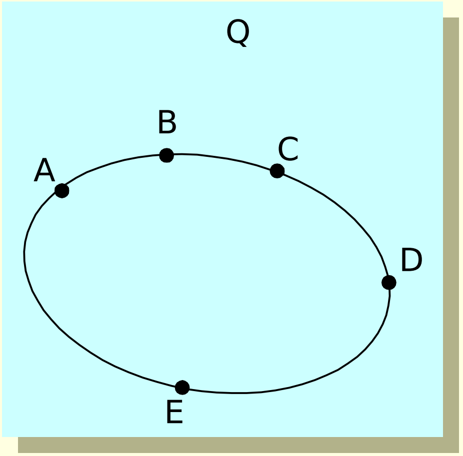



# Linear Combo of Q1 and Q2



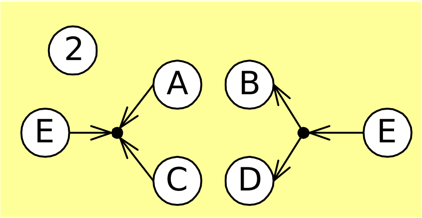

$$\mathbf{Q} = a\mathbf{Q}_1 + b\mathbf{Q}_2$$

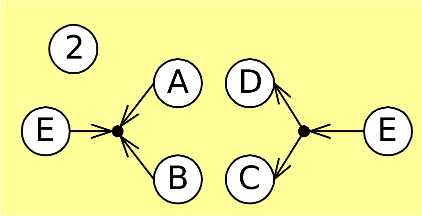
# Pick α,β to make Point E be on Q



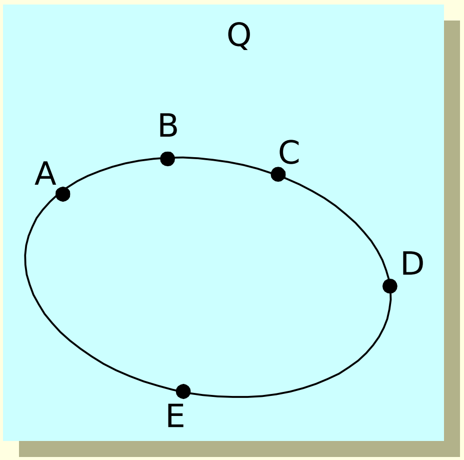

$$0 = \mathbf{E}\mathbf{Q}\mathbf{E}^T$$
$$= \mathbf{E}\left(a\mathbf{Q}_1 + b\mathbf{Q}_2\right)\mathbf{E}^T$$
$$= a\left(\mathbf{E}\mathbf{Q}_1\mathbf{E}^T\right) + b\left(\mathbf{E}\mathbf{Q}_2\mathbf{E}^T\right)$$

$$a = \mathbf{E}\mathbf{Q}_2\mathbf{E}^T$$
$$b = -\mathbf{E}\mathbf{Q}_1\mathbf{E}^T$$

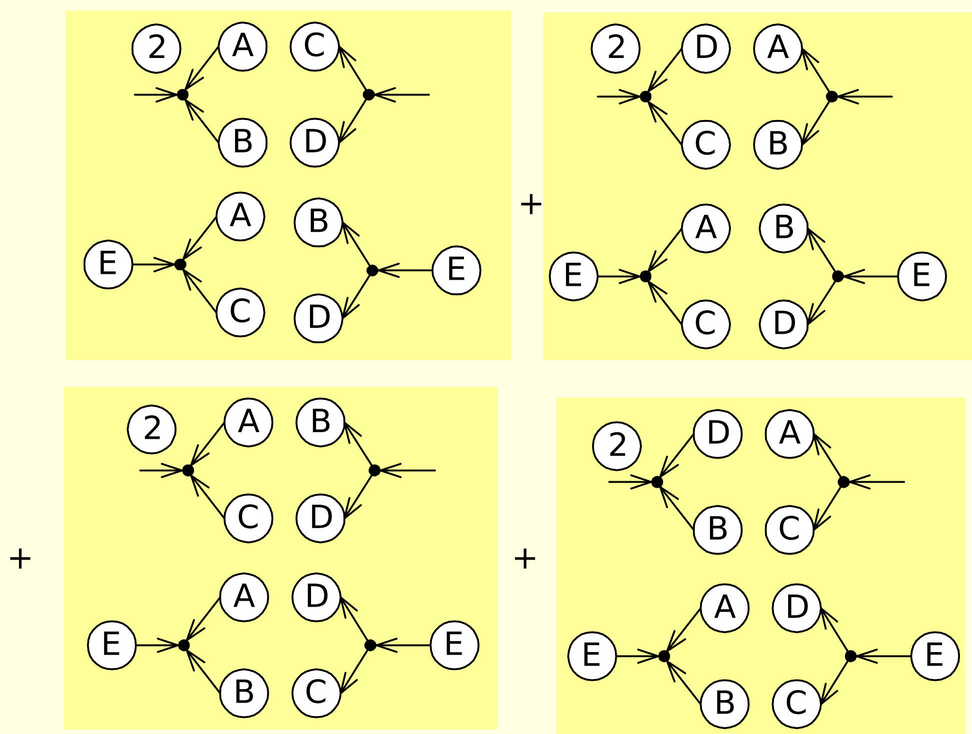
$\alpha =$ 


$\beta =$ 

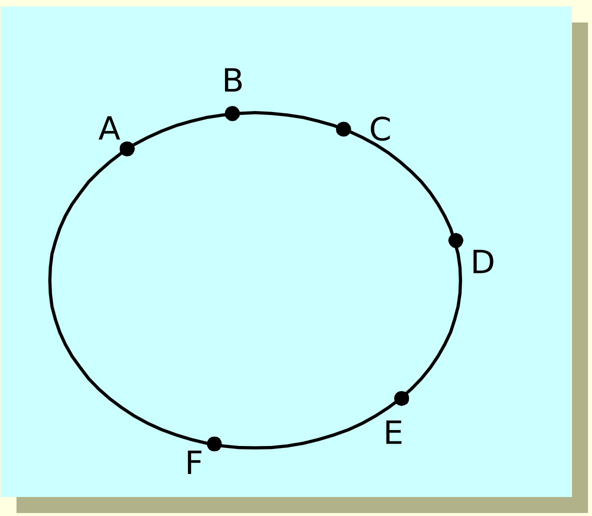
# Quadratic on 5 Points

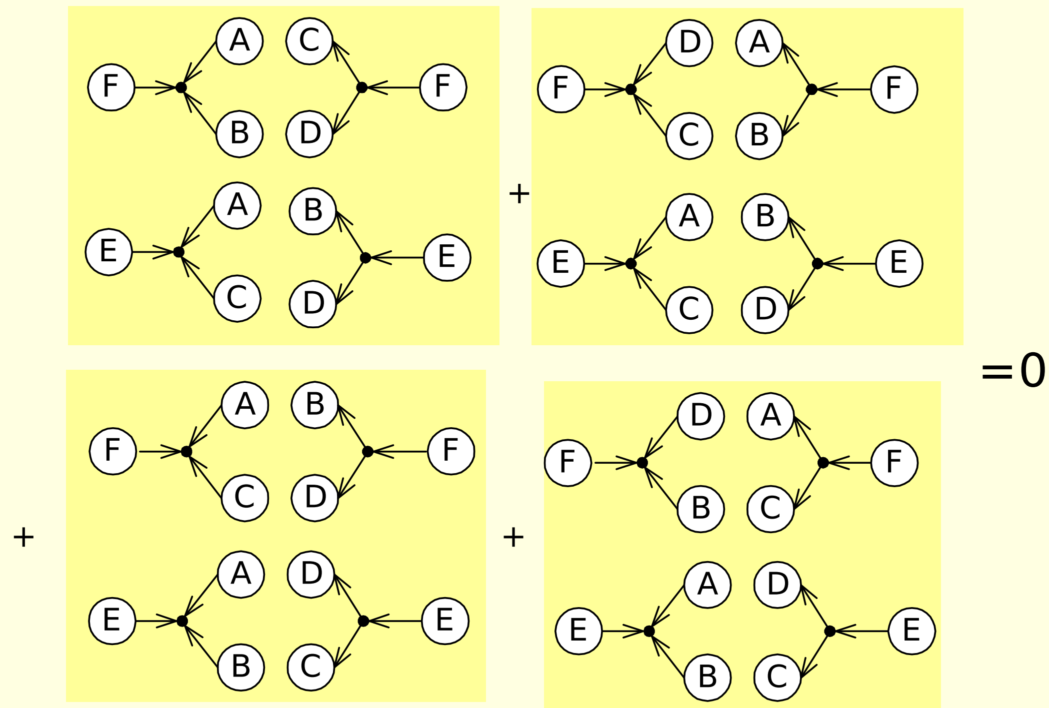



$$\mathbf{Q} = a\,\mathbf{Q}_1 + b\,\mathbf{Q}_2$$



# Six Points (ABCDEF) on Quadratic



# Pascal’s Theorem